UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
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ECE 204 *Numerical methods*

Finite-difference methods for boundary-value problems

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Introduction

- In this topic, we will
  - Describe finite-difference approximations of linear ordinary differential equations (LODEs)
  - See how this can be used to approximate solutions to boundary-value problems (BVPs)
  - Observe that this defines a system of linear equations
  - Look at examples with both constant coefficients and with variable coefficients
  - Describe implementations in MATLAB

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Linear ordinary differential equations

- In this lecture, we will focus on a technique appropriate for linear ordinary differential equations (LODEs)
  - The most general form is a linear combination of $u(x)$ and its derivatives:
    $$a_2(x)u^{(2)}(x) + a_1(x)u^{(1)}(x) + a_0(x)u(x) = g(x)$$
  - The coefficients can be functions of x
- In your calculus course, you focused on solutions to LODEs with constant coefficients:
  $$a_2u^{(2)}(x) + a_1u^{(1)}(x) + a_0u(x) = g(x)$$
  - These approximation techniques will, however, generalize

Approximating the derivative

- Previously, we saw two approximations:
  $$u^{(1)}(x) \approx \frac{-u(x-h) + u(x+h)}{2h}$$
  $$u^{(2)}(x) \approx \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}$$
  - How about substituting these two approximations into the LODE?
    $$a_2(x)u^{(2)}(x) + a_1(x)u^{(1)}(x) + a_0(x)u(x) = g(x)$$

A finite-difference method

- Thus, we go from

$$a_2(x)u^{(2)}(x) + a_1(x)u^{(1)}(x) + a_0(x)u(x) = g(x)$$

to

$$a_2(x)\left(\frac{u(x-h) - 2u(x) + u(x+h)}{h^2}\right) + a_1(x)\left(\frac{-u(x-h) + u(x+h)}{2h}\right)$$
$$+ a_0(x)u(x) \approx g(x)$$

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A finite-difference method

- Let's expand this and collect on $u(x-h)$, $u(x)$ and $u(x+h)$:

$$a_2(x)\left(\frac{u(x-h) - 2u(x) + u(x+h)}{h^2}\right) + a_1(x)\left(\frac{-u(x-h) + u(x+h)}{2h}\right)$$
$$+ a_0(x)u(x) \approx g(x)$$

$$u(x-h)\left(\frac{a_2(x)}{h^2} - \frac{a_1(x)}{2h}\right) + u(x)\left(-\frac{2a_2(x)}{h^2} + a_0(x)\right) + u(x+h)\left(\frac{a_2(x)}{h^2} + \frac{a_1(x)}{2h}\right) \approx g(x)$$

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- Finally, multiply by $2h^2$:

$$u(x-h)\left(\frac{a_2(x)}{h^2}-\frac{a_1(x)}{2h}\right)+u(x)\left(-\frac{2a_2(x)}{h^2}+a_0(x)\right)+u(x+h)\left(\frac{a_2(x)}{h^2}+\frac{a_1(x)}{2h}\right)\approx g(x)$$

$$u(x-h)\big(2a_2(x)-a_1(x)h\big)+u(x)\big(-4a_2(x)+2h^2a_0(x)\big)+u(x+h)\big(2a_2(x)+a_1(x)h\big)\approx 2g(x)h^2$$

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- Therefore, if $u(x)$ satisfies this LODE,

$$a_2(x)u^{(2)}(x)+a_1(x)u^{(1)}(x)+a_0(x)u(x)=g(x)$$

then it must also be true that

$$u(x-h)\big(2a_2(x)-a_1(x)h\big)+u(x)\big(-4a_2(x)+2h^2a_0(x)\big)+u(x+h)\big(2a_2(x)+a_1(x)h\big)\approx 2g(x)h^2$$

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A finite-difference method

Visualization

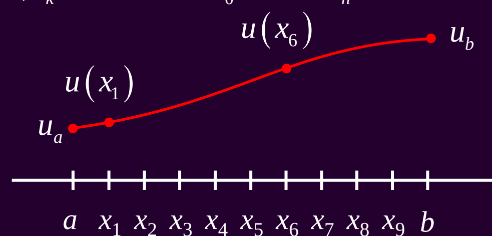
- Let's look at the problem visually:

$$a_2(x)u^{(2)}(x) + a_1(x)u^{(1)}(x) + a_0(x)u(x) = g(x)$$

$$u(a) = u_a$$

$$u(b) = u_b$$

  – Break the interval $[a, b]$ into n sub-intervals
    - Each is of width $h = \frac{b-a}{n}$
    - Thus, $x_k = a + kh$ with $x_0 = a$ and $x_n = b$



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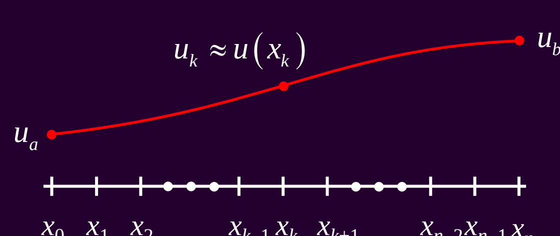
A finite-difference method

Visualization

- Let's focus on a single point x_k:
  – We don't know the value of $u(x_k)$,
    but the following equation should hold approximately true:

$$u(x_k - h)\left(2a_2(x_k) - a_1(x_k)h\right) + u(x_k)\left(-4a_2(x_k) + 2h^2 a_0(x_k)\right) + u(x_k + h)\left(2a_2(x_k) + a_1(x_k)h\right) \approx 2g(x_k)h^2$$

  – Represent our estimate of $u(x_k)$ with u_k so $u_k \approx u(x_k)$



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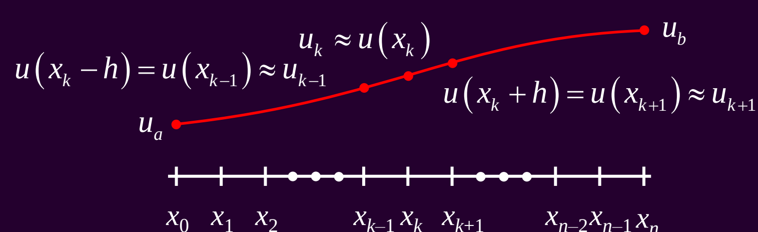
Visualization

- Note that $x_k - h = x_{k-1}$ and $x_k + h = x_{k+1}$,

  $u(x_k - h) = u(x_{k-1}) \approx u_{k-1}$

  $u(x_k + h) = u(x_{k+1}) \approx u_{k+1}$



Visualization

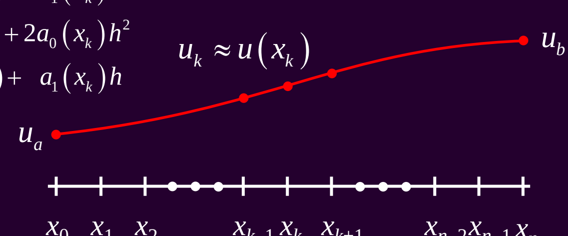
- This looks ugly, but all four functions a_2, a_1, a_0 and g as well as h are all known

$$u_{k-1}\left(2a_2(x_k) - a_1(x_k)h\right) + u_k\left(-4a_2(x_k) + 2h^2 a_0(x_k)\right) + u_{k+1}\left(2a_2(x_k) + a_1(x_k)h\right) \approx 2g(x_k)h^2$$

  – Therefore, this is a linear equation in three unknowns

$$p_k u_{k-1} + q_k u_k + r_k u_{k+1} \approx 2g(x_k)h^2$$

$$p_k = 2a_2(x_k) - a_1(x_k)h$$
$$q_k = -4a_2(x_k) + 2a_0(x_k)h^2$$
$$r_k = 2a_2(x_k) + a_1(x_k)h$$



Visualization

- There are unknowns from $k = 1, 2, \ldots, n - 1$

$$p_1 u_0 + q_1 u_1 + r_1 u_2 = 2g(x_1)h^2$$
$$p_2 u_1 + q_2 u_2 + r_2 u_3 = 2g(x_2)h^2$$
$$p_3 u_2 + q_3 u_3 + r_3 u_4 = 2g(x_2)h^2$$
$$p_4 u_3 + q_4 u_4 + r_4 u_5 = 2g(x_2)h^2$$

$$p_{n-2} u_{n-3} + q_{n-2} u_{n-2} + r_{n-2} u_{n-1} = 2g(x_{n-2})h^2$$
$$p_{n-1} u_{n-2} + q_{n-1} u_{n-1} + r_{n-1} u_n = 2g(x_{n-1})h^2$$

  – This gives $n - 1$ equations in the $n + 1$ unknowns $u_0, u_1, \ldots, u_{n-1}, u_n$

A finite-difference method

Visualization

- Fortunately, we have two boundary values, so:

$$u_0 = u_a$$
$$u_n = u_b$$

  – Thus, Equations for $k = 1$ and $k = n - 1$ may be slightly modified:

$$p_1 u_a + q_1 u_1 + r_1 u_2 = 2g(x_1)h^2$$
$$q_1 u_1 + r_1 u_2 = 2g(x_1)h^2 - p_1 u_a$$

$$p_{n-1} u_{n-2} + q_{n-1} u_{n-1} + r_{n-1} u_b = 2g(x_{n-1})h^2$$
$$p_{n-1} u_{n-2} + q_{n-1} u_{n-1} = 2g(x_{n-1})h^2 - r_{n-1} u_b$$

- Thus, we have a system of $n-1$ linear equations in $n-1$ unknowns
  - This is a tri-diagonal matrix
    - It can be solved in O(n) time, and not O(n^3) time

$$\begin{pmatrix} q_1 & r_1 & & & & & \\ p_2 & q_2 & r_2 & & & & \\ & p_3 & q_3 & r_3 & & & \\ & & p_4 & q_4 & r_4 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & p_{n-2} & q_{n-2} & r_{n-2} \\ & & & & & p_{n-1} & q_{n-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 2g(x_1)h^2 - p_1 u_a \\ 2g(x_2)h^2 \\ 2g(x_3)h^2 \\ 2g(x_4)h^2 \\ \vdots \\ 2g(x_{n-2})h^2 \\ 2g(x_{n-1})h^2 - r_{n-1} u_b \end{pmatrix}$$

- Suppose we have a LODE with constant coefficients:

$$u_{k-1}(2a_2 - a_1 h) + u_k\left(-4a_2 + 2h^2 a_0\right) + u_{k+1}(2a_2 + a_1 h) \approx 2g(x_k)h^2$$

  - Now the matrix entries are identical:

$$\begin{aligned} p &= 2a_2 - a_1 h \\ q &= -4a_2 + 2a_0 h^2 \\ r &= 2a_2 + a_1 h \end{aligned}$$

- Our matrix is now greatly simplified:
  - All entries on the diagonal, the super-diagonal and the sub-diagonal are the same

$$\begin{pmatrix} q & r & & & & & \\ p & q & r & & & & \\ & p & q & r & & & \\ & & p & q & r & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & p & q & r \\ & & & & & p & q \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 2g(x_1)h^2 - pu_a \\ 2g(x_2)h^2 \\ 2g(x_3)h^2 \\ 2g(x_4)h^2 \\ \vdots \\ 2g(x_{n-2})h^2 \\ 2g(x_{n-1})h^2 - ru_b \end{pmatrix}$$

Examples

- We will first look at both an implementation of, and an example of a BVP with constant coefficients
  - That is, a_2, a_1 and a_0 fixed real values

- Next, we will look at the implementation of, and an example of a BVP with non-constant coefficients
  - That is, a_2, a_1 and a_0 are functions of x

Constant coefficient example

```
function [xs, us] = bvpcc( ode, g, x_rng, u_bndry, n )
    h = (x_rng(2) - x_rng(1))/n;

    p =  2.0*ode(1) -     ode(2)*h;
    q = -4.0*ode(1) + 2.0*ode(3)*h^2;
    r =  2.0*ode(1) +     ode(2)*h;

    A = diag( q*ones( n - 1, 1 ) )      ...
      + diag( r*ones( n - 2, 1 ),  1 )  ...
      + diag( p*ones( n - 2, 1 ), -1 );

    xs = linspace( x_rng(1) + h, x_rng(2) - h, n - 1 )';
    v = 2.0*g( xs )*h^2;

    v(1)   = v(1)   - p*u_bndry(1);
    v(end) = v(end) - r*u_bndry(2);

    us = A \ v;
    xs = [x_rng(1);   xs; x_rng(2)];
    us = [u_bndry(1); us; u_bndry(2)];
end
```


Constant coefficient example

- Suppose we have the following BVP:

$$u^{(2)}(x) + 3u^{(1)}(x) + 2u(x) = \sin(x)$$
$$u(-1) = 1$$
$$u(1) = 2$$

- If n = 10, then h = 0.2, so

$$\begin{aligned} p &= 2a_2 - a_1 h && = 2\cdot 1 - 3\cdot 0.2 && = 1.4 \\ q &= -4a_2 + 2a_0 h^2 && = -4\cdot 1 + 2\cdot 2\cdot 0.04 && = -3.84 \\ r &= 2a_2 + a_1 h && = 2\cdot 1 - 3\cdot 0.2 && = 2.6 \end{aligned}$$

- Also, the x-values are –1, –0.8, –0.6, –0.4, –0.2, 0, 0.2, 0.4, 0.6, 0.8, 1

Constant coefficient example

- Thus, we have our system of linear equations

$$\begin{pmatrix} -3.84 & 2.6 & & & & & & & \\ 1.4 & -3.84 & 2.6 & & & & & & \\ & 1.4 & -3.84 & 2.6 & & & & & \\ & & 1.4 & -3.84 & 2.6 & & & & \\ & & & 1.4 & -3.84 & 2.6 & & & \\ & & & & 1.4 & -3.84 & 2.6 & & \\ & & & & & 1.4 & -3.84 & 2.6 & \\ & & & & & & 1.4 & -3.84 & 2.6 \\ & & & & & & & 1.4 & -3.84 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 2\sin(-0.8)0.04 - 1.4\cdot 1 \\ 2\sin(-0.6)0.04 \\ 2\sin(-0.4)0.04 \\ 2\sin(-0.2)0.04 \\ 2\sin(0)0.04 \\ 2\sin(0.2)0.04 \\ 2\sin(0.4)0.04 \\ 2\sin(0.6)0.04 \\ 2\sin(0.8)0.04 - 2.6\cdot 2 \end{pmatrix}$$

A finite-difference method

Constant coefficient example

- Solving this system of linear equations yields:

$$\mathbf{u} = \begin{pmatrix} 3.265385521974507 \\ 4.262189198888517 \\ 4.519267459652307 \\ 4.367603345860092 \\ 4.011049560817661 \\ 3.572225242052188 \\ 3.122218881076646 \\ 2.699753319940751 \\ 2.323506812667992 \end{pmatrix} \begin{matrix} \approx u(-0.8) \\ \\ \\ \\ \\ \\ \approx u(0.4) \\ \\ \\ \end{matrix}$$

A finite-difference method

Constant coefficient example

- In MATLAB, we can find this by calling:

```
>> [xs, us] = bvpcc( [1 3 2], @sin, [-1 1], [1 2], 10 );
>> u = @(x)( -0.32523483298690115734*exp( 2.0    - 2.0*x) ...
            - 0.19505820123410984121*exp(-1.0*x - 1.0)   ...
            + 0.32523483298690115734*exp(-1.0*x + 3.0)   ...
            + 0.19505820123410984121*exp(-2.0*x)         ...
            - 0.3*cos(x) + 0.1*sin(x);
```

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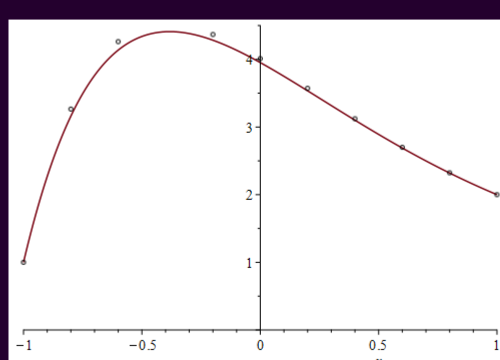
A finite-difference method

Constant coefficient example

- Here is a plot of the solution and the approximations:



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General example

```
function [xs, us] = bvp( a2, a1, a0, g, x_rng, u_bndry, n )
    h = (x_rng(2) - x_rng(1))/n;

    p = @(x)( 2.0*a2(x) -     a1(x)*h  );
    q = @(x)(-4.0*a2(x) + 2.0*a0(x)*h^2);
    r = @(x)( 2.0*a2(x) +     a1(x)*h  );

    xs = linspace( x_rng(1) + h, x_rng(2) - h, n - 1 )';

    A = zeros( n - 1, n - 1 );

    for k = 1:(n - 1)
        A(k, k) = q(xs(k));
    end

    for k = 1:(n - 2)
        A(k + 1, k    ) = p(xs(k + 1));
        A(k,     k + 1) = r(xs(k));
    end
```


General example

```
    v = 2.0*g( xs )*h^2;

    v(1)   = v(1)   - p(xs(1)  )*u_bndry(1);
    v(end) = v(end) - r(xs(end))*u_bndry(2);

    us = A \ v;
    xs = [x_rng(1);   xs; x_rng(2)];
    us = [u_bndry(1); us; u_bndry(2)];
end
```

A finite-difference method

General example

- Suppose we have the following BVP:

$$13x^2u^{(2)}(x) - 5u^{(1)}(x) + 8xu(x) = \sin(x)$$
$$u(-1) = 1$$
$$u(1) = 2$$

- If n = 10, then h = 0.2, so

$$\begin{aligned} p_k &= 2a_2(x_k) - a_1(x_k)h &&= 2\cdot 13x_k^2 - (-5)\cdot 0.2 \\ q_k &= -4a_2(x_k) + 2a_0(x_k)h^2 &&= -4\cdot 13x_k^2 + 2\cdot 8x\cdot 0.04 \\ r_k &= 2a_2(x_k) + a_1(x_k)h &&= 2\cdot 13x_k^2 + (-5)\cdot 0.2 \end{aligned}$$

- As before,
  the x-values are –1, –0.8, –0.6, –0.4, –0.2, 0, 0.2, 0.4, 0.6, 0.8, 1

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A finite-difference method

General example

- Thus, we have our system of linear equations

$$\begin{pmatrix} -33.8 & 15.6 & & & & & & & \\ 10.4 & -19.1 & 8.4 & & & & & & \\ & 5.2 & -8.6 & 3.2 & & & & & \\ & & 2.0 & -2.2 & 0.0 & & & & \\ & & & 1.0 & 0.0 & -1.0 & & & \\ & & & & 2.0 & -2.0 & 0.0 & & \\ & & & & & 5.2 & -8.1 & 3.2 & \\ & & & & & & 10.4 & -18.3 & 8.4 \\ & & & & & & & 17.6 & -32.8 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 2\sin(-0.8)0.04 - 17.6\cdot 1 \\ 2\sin(-0.6)0.04 \\ 2\sin(-0.4)0.04 \\ 2\sin(-0.2)0.04 \\ 2\sin(0)0.04 \\ 2\sin(0.2)0.04 \\ 2\sin(0.4)0.04 \\ 2\sin(0.6)0.04 \\ 2\sin(0.8)0.04 - 15.6\cdot 2 \end{pmatrix}$$

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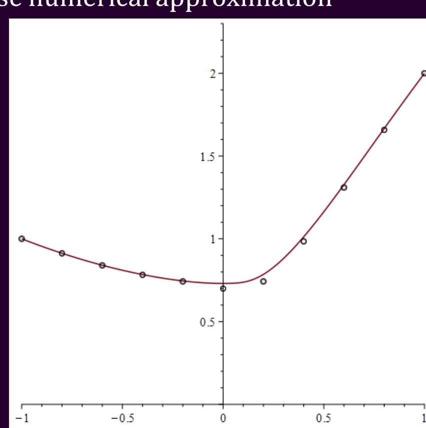
General example

- Solving this system of linear equations yields:

$$\mathbf{u} = \begin{pmatrix} 0.912226606004807 \\ 0.839422824989929 \\ 0.782355814895478 \\ 0.742692476016105 \\ 0.699139454829521 \\ 0.742692476016104 \\ 0.984619294870424 \\ 1.309756419296422 \\ 1.657919761630765 \end{pmatrix} \begin{matrix} \\ \\ \\ \approx u(-0.2) \\ \\ \\ \\ \\ \approx u(0.8) \end{matrix}$$

General example

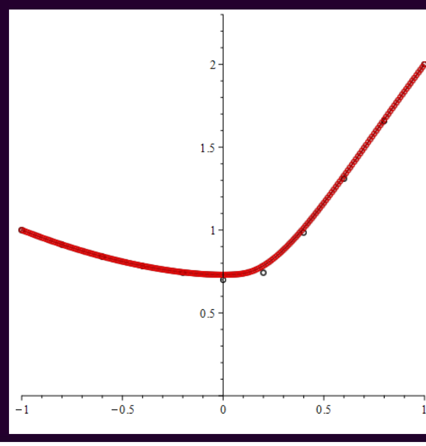
- Here is a plot of the solution and the approximations:
  - In this case, there is no exact solution, so the *solution* is actually a very precise numerical approximation



General example

- We can also use more points:

```
>> [xs, us] = bvp( @(x)(x^2*13), @(x)(-5.0), @(x)(8*x),...
                   @sin, [-1 1], [1 2], 200 )
```



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Error analysis

- Beyond the scope of this course,
  even though we are using two O(h^2) approximations of the derivative and second derivative
  - The overall error of this method is still O(h^2) as we are solving these simultaneously
  - Thus, doubling the number of intervals reduces the error by four

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Summary

- Following this topic, you now
  - Understand the idea finite difference approximations
    - For a LODE, we substitute approximations of the derivative and second derivatives
  - Know that this defines a system of linear equations
  - Understand that this solution gives the approximations at the equally spaced points between *a* and *b*
  - Are aware that if the LODE has constant coefficients, all entries on the diagonal, super-diagonal and sub-diagonal are equal, respectively
  - Have seen implementations in MATLAB

References

[1] https://en.wikipedia.org/wiki/Finite_difference_method

Acknowledgments

None so far.

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Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
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